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Charge as a manifestation of (4+1) space–time

Dexter J Booth†

Department of Mathematics, University of Victoria, Victoria, BC, Canada V8W 2Y2

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Abstract. The Reissner–Nordstrom solution to the Einstein–Maxwell field equations that represents a spherically symmetric massless charge is derived without any recourse whatsoever to Maxwellian electrodynamics. Indeed charge itself is not even postulated. It is found that the gravitational field in (3 + 1) space–time due to that which is currently accepted as classical charge may be due simply to existence within a (4 + 1) space–time.

1. Introduction

The Einstein–Maxwell field equations

$$G_{ij} = K(\overset{(m)}{T_{ij}} + \overset{(e)}{T_{ij}})$$

where

$$\overset{(m)}{T_{ij}} = \text{the energy–momentum stress tensor}$$

$$\overset{(e)}{T_{ij}} = \text{the electromagnetic energy–momentum tensor}$$

have been solved for the physical construct of a spherically symmetric mass m upon which a charge resides. This solution is the familiar Reissner–Nordstrom metric

$$\mathrm{d}s^2 = \left(1 - \frac{2Gm}{c^2 r} + \frac{\alpha}{r^2}\right) c^2\, \mathrm{d}t^2 - \left(1 - \frac{2Gm}{c^2 r} + \frac{\alpha}{r^2}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)$$

where (t, r, θ, ϕ) represent the spherical coordinates of (3 + 1) space–time (3 spatial and 1 temporal coordinates).

If we now take

$$m = 0 \qquad \text{(a 'massless' charge)}$$

then the metric becomes

$$\mathrm{d}s^2 = \left(1 + \frac{\alpha}{r^2}\right) c^2\, \mathrm{d}t^2 - \left(1 + \frac{\alpha}{r^2}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2).$$

† On leave from: The Department of Computer Studies and Mathematics, The Polytechnic, Huddersfield, HD4 4DH, England.

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Since the g_{00} component of the metric is accepted as being analogous to the Newtonian gravitational potential then it must be argued that the presence of that which we call charge produces a non-Coulombic gravitational potential; the potential obeys an inverse square law.

2. Laplace's equation

From classical Newtonian potential theory we find that in $\mathbb{R}^3$ massive bodies attract each other gravitationally with a force that is inversely proportional to the square of their separation. Performing a similar operational manoeuvre in $\mathbb{R}^2$ we find an equivalent force that is inversely proportional to the separation. Indeed, such behaviour is described by solutions to Laplace's equation.

In terms of a polar radial parameter ρ in $\mathbb{R}^2$ we find Laplace's equation as

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)f(\rho)=0$$

with solution

$$f(\rho)=\alpha+\beta\ln\rho.$$

In $\mathbb{R}^3$, Laplace's equation in terms of a spherical radial parameter r is given as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)g(r)=0$$

with solution

$$g(r)=\alpha+\beta/r.$$

As a purely mathematical exercise we can easily show that in $\mathbb{R}^4$, Laplace's equation in terms of a 'super-spherical' radial parameter R is given as

$$\frac{1}{R^3}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^3\frac{\mathrm{d}}{\mathrm{d}R}\right)h(R)=0$$

with solution

$$h(R)=\alpha+\beta/R^2.$$

In fact, a Newtonian-type theory in $\mathbb{R}^4$ would produce a force between bodies proportional to the cube of their separation distance.

One is naturally led to the question: does the 'massless' charge solution of the Einstein–Maxwell equations have any connection with $\mathbb{R}^4$?

3. (4+1) space–time

To attempt to answer the question of the last section within the confines of Einstein's gravitational theory we must extend the geometry from one of three spatial and one temporal dimensions (3+1) to one of four spatial and one temporal dimensions (4+1).

In flat (4 + 1) space–time the super-spherically symmetric metric then takes the form (Misner *et al* 1973)

$$ds^2 = c^2\, dt^2 - dR^2 - R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

$$(x^0, x^1, x^2, x^3, x^4) = (ct, R, \chi, \theta, \phi)$$

and the familiar (3 + 1) space–time of our immediate experience is achieved on the $\chi = \frac{1}{2}\pi$ hypersurface

$$ds^2 = c^2\, dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

4. Re-derivation

Consider the metric

$$ds^2 = e^{2\nu} c^2\, dt^2 - e^{-2\nu}\, dR^2 - R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

where $\nu \equiv \nu(R)$. Analysing this metric through a (4 + 1) Riemannian geometry (see the appendix) we find that the empty space–time conditions

$$R_{ij} = 0 \qquad (i, j = 0, 1, 2, 3, 4)$$

yield two equations

$$\frac{1}{R^3}\frac{d}{dR}\left(R^3\frac{d}{dR}\right) e^{2\nu} = 0 \qquad e^{2\nu}(1 + R\nu'(R)) = 1.$$

The solution to these equations is of the form

$$e^{2\nu} = 1 + k/R^2 \qquad (k \equiv \text{constant})$$

giving a metric

$$ds^2 = \left(1 + \frac{k}{R^2}\right)c^2\, dt^2 - \left(1 + \frac{k}{R^2}\right)^{-1} dR^2 - R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

which on the $\chi = \frac{1}{2}\pi$ hypersurface becomes

$$ds^2 = \left(1 + \frac{k}{r^2}\right)c^2\, dt^2 - \left(1 + \frac{k}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

If we take $k > 0$† then we have the Reissner–Nordstrom solution for massless charge.

5. Discussion

This rather curious result would appear to indicate that, from the gravitational aspect at least, charge may be a manifestation of existence within a higher dimensional structure than the one of our familiar world. The use of such a structure to incorporate electrodynamics and gravitation within a single geometry is not new, being first

† The author is indebted to the referee for pointing out the possibility of choosing $k < 0$ in the final result. The positivity of k is demanded in order to obtain correspondence with the electro-vac solution in just the same manner in which the integration constant is chosen to be negative in the Schwarzschild solution. There is no reason to anticipate attractive potentials in $\mathbb{R}^4$ in favour of repulsive potentials.

proposed by Kaluza (1921), later by Klein (1926) and currently by various authors (Veblen 1933, Cremmer and Scherk 1976, 1977, Scherk and Schwartz 1979). However, the distinction between the work in this paper and the work on Kaluza–Klein theories is that here the metric coefficients are allowed to be dependent upon the additional spatial parameter, thereby postulating an existence within a physical (4 + 1) space–time. The observed universal expansion and its prediction by Einstein's equations does support such an existential description with an additional spatial dimension beyond our immediate experience and this new result gives further credence to that view.

Appendix

Herein are listed the non-zero Christoffel symbols and the Ricci tensor components that result from the metric

$$ds^2 = e^{2\nu}c^2\,dt^2 - e^{-2\nu}\,dR^2 - R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)]$$

where

$$(x^0, x^1, x^2, x^3, x^4) \equiv (ct, R, \chi, \theta, \phi).$$

$$\Gamma^0_{01} = \nu_1 \qquad \Gamma^1_{00} = \nu_1\,e^{4\nu} \qquad \Gamma^1_{11} = -\nu_1 \qquad \Gamma^1_{22} = -R\,e^{2\nu} \qquad (\nu_1 \equiv \nu'(R))$$

$$\Gamma^1_{33} = -R\sin^2\chi\,e^{2\nu} \qquad \Gamma^1_{44} = -R\sin^2\chi\sin^2\theta\,e^{2\nu}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{R} \qquad \Gamma^2_{33} = -\sin\chi\cos\chi$$

$$\Gamma^2_{44} = -\sin\chi\cos\chi\sin^2\theta \qquad \Gamma^3_{44} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \Gamma^4_{24} = \cot\chi \qquad \Gamma^4_{34} = \cot\theta$$

$$R_{00} = \tfrac{1}{2}e^{2\nu}\,\nabla^2\,e^{2\nu} \qquad \nabla^2 \equiv \frac{1}{R^3}\frac{d}{dR}\left(R^3\frac{d}{dR}\right)$$

$$R_{0i} = 0 \qquad i \neq 0$$

$$R_{11} = -\tfrac{1}{2}e^{-2\nu}\nabla^2\,e^{2\nu}$$

$$R_{1i} = 0 \qquad i \neq 1$$

$$R_{22} = -2[e^{2\nu}(1 + \nu_1 R) - 1]$$

$$R_{2i} = 0 \qquad i \neq 2$$

$$R_{33} = -2\sin^2\chi[e^{2\nu}(1 + \nu_1 R) - 1]$$

$$R_{3i} = 0 \qquad i \neq 3$$

$$R_{44} = -2\sin^2\chi\sin^2\theta[e^{2\nu}(1 + \nu_1 R) - 1]$$

$$R_{4i} = 0 \qquad i \neq 4.$$

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